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CHEBYSHEV ESTIMATES FOR BEURLING GENERALIZED PRIME NUMBERS. I

JASSON VINDAS

ABSTRACT. We provide new sufficient conditions for Chebyshev estimates for Beurling generalized primes. It is shown that if the counting function N of a generalized number system satisfies the L^1 -condition

$$\int_1^\infty \left| \frac{N(x) - ax}{x} \right| \frac{dx}{x} < \infty$$

and $N(x) = ax + o(x/\log x)$, for some $a > 0$, then

$$0 < \liminf_{x \rightarrow \infty} \frac{\psi(x)}{x} \quad \text{and} \quad \limsup_{x \rightarrow \infty} \frac{\psi(x)}{x} < \infty$$

hold. We give an analytic proof of this result. It is based on the Wiener division theorem. Our result extends those of Diamond (Proc. Amer. Math. Soc. 39 (1973), 503–508) and Zhang (Proc. Amer. Math. Soc. 101 (1987), 205–212).

1. INTRODUCTION

This note reports on new conditions that ensure the validity of Chebyshev estimates for Beurling's generalized primes. We will considerably improve earlier results by Diamond [4] and Zhang [12]. In particular, we shall answer an open question posed by Diamond in [5, p. 10].

Let $P = \{p_k\}_{k=1}^\infty$ be a set of Beurling generalized primes, that is, a non-decreasing sequence of real numbers tending to infinity, where it is assumed $p_1 > 1$. The sequence $\{n_k\}_{k=1}^\infty$ denotes its associated set of generalized integers [1, 2]. Set further,

$$N(x) = N_P(x) = \sum_{n_k < x} 1 \quad \text{and} \quad \psi(x) = \psi_P(x) = \sum_{n_k < x} \Lambda(n_k) ,$$

where $\Lambda = \Lambda_P$ is the von Mangoldt function of the generalized number system [1]. Beurling [2] investigated the truth of the prime number theorem (PNT) in this context, i.e.,

$$\psi(x) \sim x , \quad x \rightarrow \infty .$$

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He proved that

$$(1.1) \quad N(x) = ax + O\left(\frac{x}{\log^\gamma x}\right), \quad x \rightarrow \infty \quad (a > 0),$$

where $\gamma > 3/2$, suffices for the PNT to hold. His condition is sharp: When $\gamma = 3/2$, then the PNT need not hold, as exhibited by counterexamples in [2, 3]. See [7, 10] for the most recent extensions of Beurling's PNT. Since the PNT breaks down for $\gamma \leq 3/2$, a natural question arises: Under which conditions over N do Chebyshev estimates hold true? A partial answer to this question was provided by Diamond [4], he showed that (1.1) with $\gamma > 1$ is enough to obtain Chebyshev estimates, namely,

$$(1.2) \quad 0 < \liminf_{x \rightarrow \infty} \frac{\psi(x)}{x} \quad \text{and} \quad \limsup_{x \rightarrow \infty} \frac{\psi(x)}{x} < \infty.$$

On the other hand, (1.2) is not generally true when $\gamma < 1$, as follows from an example of Hall [6].

Diamond conjectured [5] that

$$(1.3) \quad \int_1^\infty \left| \frac{N(x) - ax}{x} \right| \frac{dx}{x} < \infty, \quad \text{with } a > 0,$$

would be enough for (1.2) to hold. His conjecture turned out to be false. Kahane established the negative answer to Diamond's conjecture in [8].

Remarkably, as shown in this article, if one adds a side condition to (1.3), one can indeed achieve Chebyshev estimates. Our main goal is to prove the following theorem.

Theorem 1. *Diamond's L^1 -condition and the asymptotic behavior*

$$(1.4) \quad N(x) = ax + o\left(\frac{x}{\log x}\right), \quad x \rightarrow \infty,$$

suffice for Chebyshev estimates (1.2).

Clearly, Theorem 1 extends the result of Diamond quoted above. It should be noticed that Zhang [12] also gave an extension of Diamond's theorem. Our result includes it as a particular instance:

Corollary 1 (Zhang [12]). *The Chebyshev estimates (1.2) hold if*

$$(1.5) \quad \int_1^\infty \left(\sup_{x \leq t} \left| \frac{N(t) - at}{t} \right| \right) \frac{dx}{x} < \infty \quad (a > 0).$$

Proof. Naturally, (1.5) implies (1.3). If ω is a non-increasing and non-negative function such that $\int_1^\infty \omega(x)x^{-1}dx < \infty$, one must have $\omega(x) = o(1/\log x)$; thus, Zhang's condition (1.5) always yields (1.4). \square

We shall give a proof of Theorem 1 in Section 2. We point out that the methods of Diamond and Zhang from [4, 12] are elementary. Furthermore, Diamond has asked in [5, p. 10] whether it is possible to find an analytic approach to Chebyshev inequalities. Our proof of Theorem 1 is

non-elementary and it therefore gives an answer to Diamond's question; it uses the zeta function of the generalized number system and the Wiener division theorem [9, Chap. 2]. In addition, we make use of the operational calculus for the Laplace transform of distributions [11]. It would also be interesting to find an elementary proof. Finally, it should be mentioned that Zhang has provided another related sufficient condition for Chebyshev estimates in [13]; the author announces that is also possible to obtain substantial improvements of that result. However, we will not pursue those questions here and they will be treated elsewhere.

1.1. Notation. The Schwartz spaces $\mathcal{D}(\mathbb{R})$, $\mathcal{S}(\mathbb{R})$, $\mathcal{D}'(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R})$ are well known; we refer to [11] for their properties. If $f \in \mathcal{S}'(\mathbb{R})$ has support in $[0, \infty)$, its Laplace transform is well defined as

$$\mathcal{L}\{f; s\} = \langle f(u), e^{-su} \rangle, \quad \Re s > 0,$$

and the Fourier transform \hat{f} is the distributional boundary value of $\mathcal{L}\{f; s\}$ on $\Re s = 0$.

We use the notation H for the *Heaviside function*, it is simply the characteristic function of $(0, \infty)$.

2. PROOF OF THEOREM 1

We assume (1.3) and (1.4). Our starting point is the identity

$$(2.1) \quad \mathcal{L}\{\psi(e^u); s\} = -\frac{\zeta'(s)}{s\zeta(s)} = \frac{1}{s} \cdot \frac{-(s-1)G'(s)}{(s-1)\zeta(s)} - \frac{G(s)}{s(s-1)\zeta(s)} - \frac{1}{s} + \frac{1}{s-1},$$

where

$$G(s) := \zeta(s) - \frac{a}{s-1}.$$

We set $E_1(u) := e^{-u}N(e^u) - aH(u)$. Our assumptions (1.3) and (1.4) translate into $E_1 \in L^1(\mathbb{R})$ and $uE_1(u) = o(1)$, $u \rightarrow \infty$.

Lemma 1. *$G(s)$ extends to a continuous function on $\Re s = 1$. Consequently, $(s-1)\zeta(s)$ is continuous on $\Re s = 1$ and there exists $c > 0$ such that $it\zeta(1+it) \neq 0$ for all $t \in (-3c, 3c)$.*

Proof. Clearly,

$$(2.2) \quad G(s) = s\mathcal{L}\{E_1; s-1\} + a.$$

Taking distributional boundary values, we obtain $G(1+it) = (1+it)\hat{E}_1(t) + a$. Since $E_1 \in L^1(\mathbb{R})$, \hat{E}_1 is continuous and the assertions follow at once. \square

Set $T(u) = e^{-u}\psi(e^u)$, we must show that

$$(2.3) \quad 0 < \liminf_{u \rightarrow \infty} T(u) \quad \text{and} \quad \limsup_{u \rightarrow \infty} T(u) < \infty.$$

The next step is to use the boundary behavior of (2.1) near $s = 1$ to derive (2.3). We first study convolution averages of T .

Lemma 2. *For any fixed $\phi \in \mathcal{D}(-c, c)$,*

$$(2.4) \quad \int_{-\infty}^{\infty} T(u) \hat{\phi}(u-h) du = \int_{-\infty}^{\infty} \hat{\phi}(u) du + o(1), \quad h \rightarrow \infty.$$

Proof. Fix $\phi \in \mathcal{D}(-c, c)$. We use (2.2) to decompose (2.1) further,

$$(2.5) \quad -\frac{\zeta'(s)}{s\zeta(s)} = \frac{(s-1)\mathcal{L}\{uE_1(u); s-1\}}{(s-1)\zeta(s)} - \frac{\mathcal{L}\{E_1; s-1\}}{s\zeta(s)} - \frac{G(s)}{s(s-1)\zeta(s)} - \frac{1}{s} + \frac{1}{s-1}.$$

Set now

$$g_1(t) := \lim_{\sigma \rightarrow 1^+} \frac{(\sigma-1+it)\mathcal{L}\{uE_1(u); \sigma-1+it\}}{(\sigma-1+it)\zeta(\sigma+it)} \quad \text{in } \mathcal{S}'(\mathbb{R}),$$

and

$$g_2(t) := -\lim_{\sigma \rightarrow 1^+} \left(\frac{(\sigma-1+it)\mathcal{L}\{E_1; \sigma-1+it\} + G(\sigma+it)}{(\sigma+it)(\sigma-1+it)\zeta(\sigma+it)} + \frac{1}{\sigma+it} \right)$$

in $\mathcal{S}'(\mathbb{R})$. Taking boundary values in (2.5), we obtain $\hat{T}(t) = g_1(t) + g_2(t) + \hat{H}(t)$, an equality that must be interpreted in the sense of distributions. Recall that H is the Heaviside function. By Lemma 1, g_2 is continuous on $(-3c, 3c)$. Next, applying the Riemann-Lebesgue lemma to the continuous function $\phi(t)g_2(t)$, we conclude that

$$\begin{aligned} \int_{-\infty}^{\infty} T(u) \hat{\phi}(u-h) du &= \left\langle \hat{T}(t), e^{iht} \phi(t) \right\rangle \\ &= \int_{-\infty}^{\infty} \hat{\phi}(u) du + \left\langle g_1(t), e^{iht} \phi(t) \right\rangle + o(1). \end{aligned}$$

Thus, it is enough to show that

$$\lim_{h \rightarrow \infty} \left\langle g_1(t), e^{iht} \phi(t) \right\rangle = 0.$$

Let $M \in \mathcal{S}'(\mathbb{R})$ be the distribution supported in the interval $[0, \infty)$ that satisfies $\mathcal{L}\{M; s-1\} = ((s-1)\zeta(s))^{-1}$. Notice also that $(s-1)\mathcal{L}\{E_2; s-1\} = \mathcal{L}\{E'_2; s-1\}$, where $E_2(u) = uE_1(u) = o(1)$, so we have that $g_1 = (\widehat{E'_2} * M)$, where $*$ denotes convolution [11]. Consider an even function $\eta \in \mathcal{D}(-3c, 3c)$ such that $\eta(t) = 1$ for all $t \in (-2c, 2c)$. Clearly $\eta(t)it\zeta(1+it) \neq 0$ for all $t \in (-2c, 2c)$; moreover, it is the Fourier transform of an L^1 -function. Finally, we apply the Wiener division theorem [9, p. 88] to $\eta(t)it\zeta(1+it)$ and $\phi(t)$ and conclude the existence of $f \in L^1(\mathbb{R})$ such that

$$\hat{f}(t) = \frac{\phi(t)}{\eta(t)it\zeta(1+it)}.$$

Therefore, as $h \rightarrow \infty$,

$$\left\langle g_1(t), e^{iht} \phi(t) \right\rangle = \left\langle (\widehat{E'_2} * M)(u), \hat{\phi}(u-h) \right\rangle = (E_2 * (\hat{\eta})' * f)(h) = o(1),$$

because $E_2(u) = o(1)$ and $(\hat{\eta})' * f \in L^1(\mathbb{R})$. Thus, (2.4) has been established. \square

The estimates (2.3) follow now easily from (2.4) in Lemma 2. Choose $\phi \in \mathcal{D}(-c, c)$ in (2.4) such that $\hat{\phi}$ is non-negative. Using the fact that $\psi(e^u)$ is non-decreasing, we have that $e^{-u}T(h) \leq T(u+h)$ whenever u and h are positive, setting $C_1 = \int_0^\infty e^{-u}\hat{\phi}(u)du > 0$,

$$T(h) = C_1^{-1} \int_0^\infty e^{-u}T(h)\hat{\phi}(u)du \leq C_1^{-1} \int_0^\infty T(u+h)\hat{\phi}(u)du = O(1) \leq C_2 ,$$

for some constant $C_2 > 0$. Fix now $A > 0$; observe that if $u \leq A$, then $T(h) \geq e^{u-A}T(h-A+u)$, and hence

$$\begin{aligned} \liminf_{h \rightarrow \infty} T(h) &\geq \frac{e^{-A}}{\int_{-A}^A e^{-u}\hat{\phi}(u)du} \liminf_{h \rightarrow \infty} \int_{-A}^A T(h-A+u)\hat{\phi}(u)du \\ &= \frac{e^{-A}}{\int_{-A}^A e^{-u}\hat{\phi}(u)du} \liminf_{h \rightarrow \infty} \left(\int_{-\infty}^\infty - \int_{|u| \geq A} \right) T(h-A+u)\hat{\phi}(u)du \\ &\geq \frac{e^{-A}}{\int_{-A}^A e^{-u}\hat{\phi}(u)du} \left(\int_{-\infty}^\infty \hat{\phi}(u)du - C_2 \int_{|u| \geq A} \hat{\phi}(u)du \right) . \end{aligned}$$

It remains to choose A so large that $\int_{-\infty}^\infty \hat{\phi}(u)du - C_2 \int_{|u| \geq A} \hat{\phi}(u)du > 0$. The proof is complete.

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